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Schrödinger lumps

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Abstract. We give a systematic method to construct exact non-spreading wavepacked solutions (lumps) of the Schrödinger equation for a charged particle in an electromagnetic field and of the nonlinear Schrödinger equation. The method yields new kinds of accelerating lumps with both rotational and translational motion, in contrast to the more familiar soliton solutions which have constant velocity. In particular, we find new normalisable coherent states for the harmonic oscillator and new normalisable lumps for a charged particle in a constant magnetic field plus an electric field with axial symmetry. The method can also be applied to relativisitc wave equations and classical field theories to find their lump solutions.

1. Introduction

A fundamental difficulty in obtaining lump or particle-like solutions of the Schrödinger equation was first pointed out by Lorentz (1926) (for a textbook discussion see eg Messiah (1961)): 'a wave packet can never stay together and remain confined to a small volume in the long run'. This difficulty, related to the linear and dispersive nature of the equation, was one of the reasons which led de Broglie (1960, ch 18) to introduce a nonlinear wave equation. A celebrated exception noted by Schrödinger (Schiff 1955, Schrödinger 1926) is the following exact non-spreading solution of the Schrödinger equation for a particle of mass m in a one-dimensional simple harmonic potential $\frac{1}{2}m\omega^2x^2$

$$\psi(x, t)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}[x-a\cos(\omega t)]^2-\mathrm{i}\phi(x, t)\right), \tag{1}$$

where a is real and

$$\phi(x, t)\equiv\tfrac{1}{2}\omega t+(m\omega a/\hbar)\sin(\omega t)[x-\tfrac{1}{2}a\cos(\omega t)]. \tag{2}$$

Clearly the wavepacket for $|\psi|^2$ oscillates without distortion. Recently such solutions have been discovered for the free particle Schrödinger equation by Ignatovich (1978) and Berry and Balazs (1979). Especially remarkable is (Berry and Balazs 1979) an 'accelerating wavepacket' solution for a free particle,

$$\psi(x, t)=\mathrm{Ai}\left[\frac{B}{\hbar^{2/3}}\left(x-\frac{B^3t^2}{4m^2}\right)\right]\exp\left[\mathrm{i}\frac{B^3t}{2m\hbar}\left(x-\frac{B^3t^2}{6m^2}\right)\right], \tag{3}$$

where Ai denotes the Airy function and B is a constant. Berry and Balazs (1979) have also obtained similar solutions in a time-dependent potential $xF(t)$. Among these lumps only the Schrödinger solution (1) is normalisable.

We report the results of a systematic search for lump solutions of the Schrödinger equation for a charged particle experiencing an arbitrary time-dependent electromagnetic field in addition to a charge-independent scalar potential. The class of electromagnetic fields and scalar potentials for which lump solutions can exist is derived. In addition to recovering the few lump solutions already known, we find a variety of new solutions. Especially promising for future applications are the new coherent states of the harmonic oscillator, and normalisable lumps with the novel feature of rotational motion for a charged particle in a constant magnetic field plus an electric field with axial symmetry around the magnetic field direction.

It is well known (Scott *et al* 1973) that the nonlinear Schrödinger equation has 'envelope soliton' solutions (lumps in the present language). The present work clarifies under what conditions the linear Schrödinger equation can have such solutions, and why so few of them have been noticed before: it also suggests the possibility of obtaining new kinds of lump solutions for the nonlinear equations of physics, especially for the nonlinear Schrödinger equation.

In comparing the existence of lumps for the linear and nonlinear Schrödinger equations the following general remarks are relevant. (i) Irrespective of normalisability or probability density interpretation of $|\psi(x, t)|^2$, it is of special interest to ask for a shape-preserving or lump property for $|\psi|^2$ because of its gauge invariance. (ii) The differential equations obeyed by $|\psi|^2$ are nonlinear even when ψ obeys the linear Schrödinger equation.

2. Definition

A solution $\psi(\boldsymbol{x}, t)$ will be called a lump if $|\psi|^2$ propagates in time without distortion except for a possible translation and rotation. For a lump we may write

$$\psi = R(\boldsymbol{\xi}) \exp[\mathrm{i}S(\boldsymbol{\xi}, t)/\hbar], \tag{4}$$

where R and S are real and

$$\xi_i \equiv U_{ij}(t)(x_j - a_j(t)), \qquad i, j = 1, 2, 3. \tag{5}$$

Here the $a_i(t)$ are arbitrary real functions and $U(t)$ is a real orthogonal rotation matrix,

$$U^{\mathrm{T}}U = UU^{\mathrm{T}} = 1. \tag{6}$$

If R is a constant independent of $\boldsymbol{\xi}$, or if the $\boldsymbol{a}(t)$ are constants and $U(t)$ equals the identity matrix, we obtain trivial, i.e. stationary, lumps. We shall seek only non-trivial lumps.

This definition differs from the usual definition of solitary waves (Scott *et al* 1973) in allowing $\boldsymbol{a}(t)$ to be general (rather than linear) functions and in the presence of the rotation matrix $U(t)$. The definition makes no reference to the superposition principle or to square integrability and is appropriate also for equations other than the Schrödinger equation, e.g. the nonlinear Schrödinger equation. In relativistic cases such as the Klein–Gordon equation, it may be appropriate to replace equation (4) by the statement that the charge density depends on $\boldsymbol{\xi}$ only.

3. Method

The Schrödinger equation for a charged particle in an electromagnetic field plus a charge independent potential V_0 is

$$i\hbar\frac{\partial\psi}{\partial t}=\left[e\phi+V_0+\frac{1}{2m}\left(-i\hbar\nabla-\frac{e\boldsymbol{A}}{c}\right)^2\right]\psi, \tag{7}$$

where ϕ and $\boldsymbol{A}$ denote respectively the scalar and vector electromagnetic potentials, e and m the charge and mass of the particle. Substituting (4) and using only the reality of R and S, the Schrödinger equation is seen to be equivalent to the hydrodynamical-type equations (Madelung 1926, de Broglie 1927, 1960)

$$\partial\rho/\partial t+\operatorname{div}(\rho\boldsymbol{v})=0, \qquad \rho\equiv R^2, \tag{8}$$

$$-\frac{\partial S}{\partial t}=e\phi+V_0+\tfrac{1}{2}mv^2-\frac{\hbar^2}{2m}\frac{1}{R}\nabla^2 R, \tag{9}$$

where

$$\boldsymbol{v}\equiv\boldsymbol{j}/\rho=(1/m)[\nabla S-(e/c)\boldsymbol{A}], \tag{10}$$

$$\boldsymbol{j}\equiv(1/m)\operatorname{Re}[\psi^*(-i\hbar\nabla-e\boldsymbol{A}/c)\psi]. \tag{11}$$

The gradient and curl of equations (9) and (10) respectively yield

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)m\boldsymbol{v}=e\boldsymbol{E}-\nabla V_0+\frac{e}{c}\boldsymbol{v}\times\boldsymbol{B}+\frac{\hbar^2}{2m}\nabla\left(\frac{1}{R}\nabla^2 R\right) \tag{12}$$

and

$$\operatorname{curl}\boldsymbol{v}=-(e/mc)\boldsymbol{B}, \tag{13}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields. Equations (8), (12) and (13) constitute a gauge-independent formulation of the Schrödinger equation. Equation (12) resembles the classical Lorentz-force equation, except for the last term which may be called the quantum force. Except for this term equations (8), (12) and (13) describe motion in an external electromagnetic field of a classical fluid of non-interacting particles of charge e, mass m, local density ρ and local velocity $\boldsymbol{v}$. These equations for ρ are nonlinear, though ψ obeys a linear equation.

3.1. Lump velocity versus hydrodynamical velocity

In seeking lump solutions the hydrodynamical picture tempts one to postulate that the lump-velocity $\boldsymbol{v}_{\text{lump}}$ equals the hydrodynamic velocity $\boldsymbol{v}$. In fact such an equality is a corner stone of de Broglie's (1960) interpretation of wave mechanics. The equality holds also for all lump solutions hitherto discovered (Schiff 1955, Schrödinger 1926, Ignatovich 1978, Berry and Balazs 1979). Nevertheless we show that the Schrödinger equation forces a relation between the two velocities less restrictive than their equality, and we give explicit construction also of lumps of velocity different from $\boldsymbol{v}$.

The definitions (4)–(6) imply that the lump velocity is

$$(v_{\text{lump}})_i\equiv(\partial x_i/\partial t)_\xi=\dot{a}_i(t)-\varepsilon_{ik}(t)(x_k-a_k(t)), \tag{14}$$

where $\varepsilon(t)$ is the antisymmetric matrix,

$$\varepsilon(t)\equiv U^{\mathrm{T}}(t)\dot{U}(t), \tag{15}$$

and the dots over $a(t)$ and $U(t)$ denote time derivatives. The continuity equation (8) expressed in terms of $\boldsymbol{\xi}$ and t becomes, on using (14) and (15),

$$(\partial/\partial\xi_i)_t[\rho(\boldsymbol{\xi})U_{ij}(t)(v_j-(v_{\text{lump}})_j)]=0. \tag{16}$$

The square bracket above must therefore be the curl with respect to $\boldsymbol{\xi}$ of some vector $\boldsymbol{C}(\boldsymbol{\xi}, t)$. Hence

$$v_l-(v_{\text{lump}})_l=(1/\rho(\boldsymbol{\xi}))\,U^T_{li}(t)\varepsilon_{ijk}(\partial/\partial\xi_j)_t C_k(\boldsymbol{\xi}, t). \tag{17}$$

In the special case of one space dimension equation (16) implies

$$v-v_{\text{lump}}=b(t)/\rho(\xi), \tag{18}$$

where $b(t)$ is an arbitrary function of time. We see that the continuity equation places a non-trivial restriction on $\boldsymbol{v}-\boldsymbol{v}_{\text{lump}}$, but allows it to be non-zero.

The classes of V_0, $\boldsymbol{E}$ and $\boldsymbol{B}$ for which lumps exist are now obtained by substituting (17) (or (18)) into equations (12) and (13) and requiring a solution $R=R(\boldsymbol{\xi})$ to exist. The gauge-dependent phase S is detained by integrating equation (10). We consider first the case of one space dimension.

4. Lumps in one space dimension

We have

$$\boldsymbol{B}=0, \qquad U(t)=1, \qquad \varepsilon(t)=0, \qquad \xi=x-a(t), \tag{19}$$

$$v_{\text{lump}}=\dot a(t), \qquad v=\dot a(t)+b(t)/\rho(\xi). \tag{20}$$

Substituting (20) in (12), we obtain the class of 'allowed' electric fields E and potentials V_0 for which lump solutions exist:

$$eE(x,t)-\frac{\partial}{\partial x}V_0(x,t)$$
$$=m\ddot a(t)-\frac{\hbar^2}{2m}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{R''(\xi)}{R(\xi)}+\frac{m}{R^2(\xi)}\left(\dot b(t)-2b^2(t)\frac{R'(\xi)}{R^3(\xi)}\right). \tag{21}$$

The class of allowed fields is parametrised by three arbitrary functions $a(t)$, $b(t)$ and $R(\xi)$. Any choice of these functions yields a lump solution of the Schrödinger equation corresponding to a field given by (21). The phase of the wavefunction is easily calculated from equation (10). A large variety of lumps is thus obtained.

For a fixed ξ (say $\xi=0$), equation (21) gives the equation of motion of the centre of the lump; this equation differs from the classical equation due to the quantum force term and the terms depending on $b(t)$. For a fixed t, equation (21) determines $R(\xi)$, i.e. the shape of the lump.

In a physical problem where $E(x, t)$ and $V_0(x, t)$ are specified *a priori*, the chance of equation (21) being satisfied for some $a(t)$, $b(t)$, $R(\xi)$ is remote. This partly explains why so few lumps have been previously encountered. However, the lack of a systematic approach has also prevented the discovery of lumps, as will be evident from the new lump solutions obtained here.

4.1. *Previously known lumps*

Let us first remark that we recover very simply all the previously known lumps. They correspond to

$$b(t)=0, \qquad eE-\nabla V_0(x,t)=C_1(t)+xC_2, \tag{22}$$

with $C_1=C_2=0$ for the free particle, $C_1(t)\neq 0$, $C_2=0$ for an electric field depending on time only, and $C_1=0$, $C_2\neq 0$ for a simple harmonic potential. For consistency of (21) and (22) we need

$$m\ddot{a}(t)=C_1(t)+C_2a(t)+C_3 \tag{23}$$

and

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{R''(\xi)}{R(\xi)}=C_2\xi-C_3, \tag{24}$$

where C_3 is a real constant. Note that, for a free particle, C_3 just equals the quantum force and allows the accelerating Airy lump quoted before. The shape of $R(\xi)$ is given by Airy's differential equation both for the free particle and for the time-dependent electric field case, since $C_1(t)$ enters only the equation of motion (23). For a simple harmonic oscillator of frequency ω, $C_2=-m\omega^2$, and the modulus of the original Schrödinger lump (1) is readily seen to satisfy (24).

4.2. *New lumps and new coherent states of the harmonic oscillator*

Now comes a surprise. There *are* other lumps for the simple harmonic oscillator with $b(t)=0$. Integrating equation (24) once, we obtain Weber's differential equation whose solutions are well known to be the parabolic cylinder functions (Whittaker and Watson 1963). Thus

$$R(\xi)=C_4D_n(z)+C_5\Gamma(-n)[D_n(-z)-\cos(n\pi)D_n(z)]/(2\pi)^{1/2}, \tag{25}$$

where

$$z=(2m\omega/\hbar)^{1/2}(\xi+C_3/m\omega^2), \tag{26}$$

C_4, C_5 and n are arbitrary real constants and the $D_n(z)$ denote parabolic cylinder functions. Further,

$$a(t)=C_3/m\omega^2+C_6\cos(\omega t+\phi), \tag{27}$$

C_6 and ϕ being real constants. The solution (25) is in general not square integrable. However, for $C_5=0$ and n integral, we obtain

$$R(\xi)=C_7H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\left(\xi+\frac{C_3}{m\omega^2}\right)\right]\exp\left[-\frac{1}{2}\frac{m\omega}{\hbar}\left(\xi+\frac{C_3}{m\omega^2}\right)^2\right], \tag{28}$$

which are (apart from the phase factor $\exp(\mathrm{i}S/\hbar)$) just the oscillator eigenfunctions with x replaced by $x-C_6\cos(\omega t+\phi)$. The lumps (25) and the square integrable lumps (28) for $n=1,2,3,\ldots$ are believed to be new. The latter represent square integrable, shape preserving wavepackets with classical motion, and hence may be called coherent states, in analogy to the $n=0$ Schrödinger lump, equation (1). The new coherent states have an uncertainty product $\Delta x\,\Delta p=(n+\frac{1}{2})\hbar$, showing that minimum uncertainty is not required for coherence.

Further, for the new case $C_1(t) \neq 0$ and $C_2 \neq 0$ with both a time-dependent electric field and a simple harmonic oscillator potential, lumps are present. Their shape is given by equations (25), (26) and (28), and their motion by the well known forced oscillator equation (23).

4.3. *The general case* $b(t) \neq 0$

To obtain a glimpse of how restrictive the class of allowed fields is, we seek allowed fields of the simple form

$$eE(x, t) - \partial V_0/\partial x(x, t) = C_1(t) + F(x), \tag{29}$$

most frequently encountered in physics. Substituting this in (21) and taking $(\partial/\partial t)_x$, we obtain

$$\frac{\hbar^2}{2m^2}\left(\frac{R''(\xi)}{R(\xi)}\right)'' + \frac{m\dddot{a} - \dot{C}_1}{m\dot{a}} = \frac{b^2(t)}{2}(u^2)'' + \dot{b}u' - \frac{u}{\dot{a}}(\ddot{b} + 2b\dot{b}u') \tag{30}$$

where

$$u = u(\xi) \equiv 1/R^2(\xi). \tag{31}$$

Consider first the possibility

$$\dot{b}(t) = 0, \qquad b(t) = b. \tag{32}$$

Then equation (30) requires a function of ξ alone to equal a function of t alone, and hence both must be constants,

$$\frac{m\dddot{a} - \dot{C}_1}{m\dot{a}} = \left(\frac{b^2}{2R^4(\xi)} - \frac{\hbar^2}{2m^2}\frac{R''(\xi)}{R(\xi)}\right)'' = \frac{C_2}{m}. \tag{33}$$

Integration yields the same equation for $a(t)$ as (23), but for the shape of the lump the nonlinear equation

$$-(\hbar^2/2m^2)R''(\xi) + b^2/2R^3(\xi) = (1/2m)(C_2\xi^2 - 2C_3\xi - 2C_8)R(\xi). \tag{34}$$

Equation (21) now yields the remarkable result that the corresponding allowed field *must* have

$$F(x) = C_2 x, \qquad C_1(t) \text{ arbitrary}. \tag{35}$$

Explicit solution of the nonlinear equation (34) is made possible because of the following elementary theorem (Pinney 1950, Lewis 1967, equation (6)).

Let $y_1(x)$, $y_2(x)$ be two solutions of the equation $y''(x) + g(x)y(x) = 0$, with Wronskian $y_1y_2' - y_2y_1' = 1$. Then the equation

$$Y''(x) + g(x)Y(x) = W^2/(Y(x))^3 \tag{36}$$

has the general solution

$$Y(x) = \pm[W^2(y_1^2/I) + I(Jy_1 \pm y_2)^2]^{1/2} \tag{37}$$

where I, J are arbitrary constants.

Applying the theorem to equation (34) with $C_2 = -m\omega^2$, $W^2 = mb^2/(2\hbar\omega)$ and z given by equation (26), we find that the general lump with $b(t) = b$ is given by equation (37) with $Y(x) \to R(\xi)$, with y_1, y_2 denoting respectively the parabolic cylinder functions multiplying C_4 and C_5 in equation (25), and with n, I, J arbitrary real constants. None of these solutions with $b \neq 0$ is square integrable.

Lumps with $\dot{b}(t) \neq 0$

Detailed calculations in this case are involved. We prove (appendix 1) that for $C_1(t) = 0$, the only such lump occurs for the Morse potential. Explicitly,

$$eE - \nabla V_0 = C_9 m \exp(C_9 x + C_{10} + C_{11})[C_{12} + \exp(C_9 x + C_{10} + C_{11})], \tag{38}$$

$$a(t) = C_{12} t + C_{13}, \qquad b(t) = \exp(C_9 a(t) + C_{11}), \tag{39}$$

$$v(t) = C_{12} + b(t)/R^2(\xi), \qquad R(\xi) = \exp[-\tfrac{1}{2}(C_9 \xi + C_{10})], \tag{40}$$

where $C_9, \ldots, C_{13}$ are arbitrary real constants.

We have demonstrated the existence of lumps of non-hydrodynamical velocity. For three dimensions, in order to concentrate on several new physical features, we discuss the simpler case of lumps of hydrodynamical velocity.

5. Three-dimensional lumps of hydrodynamical velocity

The choice $\boldsymbol{v} = \boldsymbol{v}_{\text{lump}}$ satisfies (17) for $\boldsymbol{C} = 0$. Equations (13) and (14) now yield

$$B_i = -(mc/e)\varepsilon_{ijk}\varepsilon_{jk}(t), \tag{41}$$

i.e. for a lump solution the magnetic field can depend on time, but *not* on space coordinates. From Maxwell's equations the corresponding electric field must be of the form

$$e\boldsymbol{E}(\boldsymbol{x}, t) - \nabla V_0(\boldsymbol{x}, t) = m(\boldsymbol{x} - \boldsymbol{a}(t)) \times \dot{\boldsymbol{\omega}}(t) - \nabla V(\boldsymbol{x}, t) \tag{42}$$

where $V(x, t)$ is an arbitrary real function and

$$\boldsymbol{\omega}(t) \equiv (e/2mc)\boldsymbol{B}(t), \qquad \omega \equiv |\omega|. \tag{43}$$

Substituting

$$\boldsymbol{v} = \boldsymbol{v}_{\text{lump}} = \dot{\boldsymbol{a}}(t) + (\boldsymbol{x} - \boldsymbol{a}(t)) \times \boldsymbol{\omega}(t), \tag{44}$$

into equation (12), we find the set of allowed $V(x, t)$ and hence of allowed electric fields:

$$V(\boldsymbol{x}, t) = (\hbar^2/2m)(1/R(\boldsymbol{\xi}))\nabla_{\xi}^2 R(\boldsymbol{\xi}) - \tfrac{1}{2}m\omega^2(t)\boldsymbol{\xi}^2 + f(t) + \tfrac{1}{2}m(U_{ij}(t)\omega_j(t)\xi_i)^2 - m\xi_i U_{ij}(t)(\ddot{\boldsymbol{a}}(t) - 2\dot{\boldsymbol{a}}(t) \times \boldsymbol{\omega}(t))_j, \tag{45}$$

where $f(t)$ is an arbitrary function. Equations (42) and (45) are the three-dimensional analogue of equation (21). The allowed electric and magnetic fields are parametrised by $U(t)$, $\boldsymbol{a}(t)$ and $R(\boldsymbol{\xi})$; any choice of them, with V given by (45), leads to a lump.

The equation of motion of the centre of the lump, $\boldsymbol{\xi} = 0$ ($\boldsymbol{x} = \boldsymbol{a}(t)$), has a relatively transparent form,

$$m\ddot{a}_i(t) = eE_i(\boldsymbol{a}, t) - \nabla_i V_0(\boldsymbol{a}, t) + \frac{e}{c}\dot{\boldsymbol{a}}(t) \times \boldsymbol{B}(t) + \frac{\hbar^2}{2m}\left(\frac{\partial}{\partial \xi_j}\right)_t \left(\frac{1}{R(\boldsymbol{\xi})}\nabla_{\xi}^2 R(\boldsymbol{\xi})\right)_{\xi=0} U_{ji}(t), \tag{46}$$

whereas the shape $R(\boldsymbol{\xi})$ is to be calculated from (45) with t fixed.

Very special space–time dependence characterises the allowed electric and magnetic fields. We give some examples of physically interesting lumps for time-independent fields.

5.1. Rotating lump: charged particle in a constant magnetic field plus electric field with axial symmetry around the magnetic field direction

For a space–time independent magnetic field in the z direction, we have $\boldsymbol{\omega} = \omega\hat{z}$ and

$$U(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{47}$$

Clearly equation (45) is consistent for the time-independent potential

$$V = g(x^2+y^2, z) = \tfrac{1}{2}m\omega^2(x^2+y^2), \qquad \boldsymbol{a} = 0, \tag{48}$$

where g is an arbitrary function invariant under rotations around the z axis. $R(\boldsymbol{\xi})$ satisfies the linear differential equation

$$(\hbar^2/2m)\nabla_\xi^2 R(\boldsymbol{\xi}) = g(\xi_1^2+\xi_2^2, \xi_3)R(\boldsymbol{\xi}). \tag{49}$$

For example, if

$$g = g(\boldsymbol{\xi}^2) = g(r^2) = -\lambda/r - E, \tag{50}$$

square integrable lumps are obtained for

$$E = -(m\lambda^2/2\hbar^2)(1/n^2), \qquad n = 1, 2, 3, \ldots, \tag{51}$$

with R being given by

$$R(\boldsymbol{\xi}) = \mathrm{Re}\sum_{l=0}^{n-1}\sum_{m'=-l}^{l} C_{lm'}R_{nlm'}(\boldsymbol{\xi}), \tag{52}$$

where $R_{nlm'}(\boldsymbol{r})$ are Coulomb eigenfunctions for the potential $(-\lambda/r)$, and $C_{lm'}$ arbitrary complex constants. The all important difference from the usual energy eigenfunctions is that

$$\boldsymbol{r} \to \boldsymbol{\xi} = r(\sin\theta\cos(\omega t+\phi), \sin\theta\sin(\omega t+\phi), \cos\theta) \tag{53}$$

at a point (r, θ, ϕ) so that equation (52) describes a non-trivial rotating lump whenever $m' \neq 0$ terms are present in the summation (52). This rotation is in accord with classical motion; however the lump is stationary in the z direction in spite of a non-zero electric field in that direction. The electric field is crucial in obtaining the square integrable lumps (52). Other rotating lumps corresponding to other choices of $g(x^2+y^2, z)$ may be similarly obtained.

5.2. Translating lump; nonharmonic potentials

Equation (45) can be made consistent for zero magnetic field, with

$$V = \boldsymbol{C}\cdot\boldsymbol{x} + \tfrac{1}{2}m(\nu_1^2x_1^2+\nu_2^2x_2^2) + h(x_3), \tag{54}$$

where C, ν_1^2, ν_2^2 are real constants and $h(x_3)$ an arbitrary real function. For example, we may choose

$$\ddot{a}_1(t) = \mu_1 - \nu_1^2a_1(t), \qquad \ddot{a}_2 = \mu_2 - \nu_2^2a_2, \qquad a_3 = 0, \tag{55}$$

$$f(t) = \boldsymbol{C}\cdot\boldsymbol{a}(t) + \tfrac{1}{2}m(\nu_1^2a_1^2(t)+\nu_2^2a_2^2(t)), \tag{56}$$

where μ_1 and μ_2 are arbitrary constants. Then the shape of the corresponding lump is obtained from a solution of the linear equation

$$-(\hbar^2/2m)\nabla_\xi^2 R(\boldsymbol{\xi})+R(\boldsymbol{\xi})[\boldsymbol{C}\cdot\boldsymbol{\xi}+\tfrac{1}{2}m(\nu_1^2\xi_1^2+\nu_2^2\xi_2^2+2\mu_1\xi_1+2\mu_2\xi_2)+h(\xi_3)]=0. \tag{57}$$

Similarly,

$$V=\boldsymbol{C}\cdot\boldsymbol{x}+\tfrac{1}{2}m\nu_1^2x_1^2+h(x_2,x_3), \tag{58}$$

with $h(x_2, x_3)$ an arbitrary real function, yields a lump with

$$\ddot{a}_1(t)=\mu_1-\nu_1^2a_1(t), \qquad a_2=a_3=0, \tag{59}$$

$$f(t)=\boldsymbol{C}\cdot\boldsymbol{a}(t)+\tfrac{1}{2}m\nu_1^2a_1^2(t), \tag{60}$$

$$-(\hbar^2/2m)\nabla_\xi^2R(\boldsymbol{\xi})+[\boldsymbol{C}\cdot\boldsymbol{\xi}+\tfrac{1}{2}m\nu_1^2\xi_1^2+m\mu_1\xi_1+h(\xi_2,\xi_3)]R(\boldsymbol{\xi})=0. \tag{61}$$

Equations (54) and (58) are examples of non-harmonic potentials for which purely translational lumps exist. However, in each case, the arbitrary function h depends only on coordinates along which the lump does not move (see equations (55) and (59)). These lumps are normalisable whenever the solution $R(\boldsymbol{\xi})$ to equation (57) or (61) is square integrable. The three-dimensional forced oscillator,

$$V=\boldsymbol{C}\cdot\boldsymbol{x}+\tfrac{1}{2}m\sum_{i=1}^{3}\nu_i^2x_i^2+h, \tag{62}$$

with h a real constant, has lump solutions with

$$\ddot{a}_i(t)=\mu_i-\nu_i^2a_i(t), \qquad i=1,2,3, \tag{63}$$

$$f(t)=\boldsymbol{C}\cdot\boldsymbol{a}(t)+\tfrac{1}{2}m\sum_{i=1}^{3}\nu_i^2a_i^2(t), \tag{64}$$

$$-\frac{\hbar^2}{2m}\nabla_\xi^2R(\boldsymbol{\xi})+\left(\boldsymbol{C}\cdot\boldsymbol{\xi}+\frac{m}{2}\sum_{i=1}^{3}(\nu_i^2\xi_i^2+2\mu_i\xi_i)+h\right)R(\boldsymbol{\xi})=0, \tag{65}$$

where the μ_i are arbitrary real constants. The solution $R(\boldsymbol{\xi})$ factorises as $R_1(\xi_1)R_2(\xi_2)R_3(\xi_3)$ where each factor obeys a differential equation of the form (24) whose solutions have already been discussed in detail.

5.3. Lumps with rotational and translational motion

Consider again the case of a space–time independent magnetic field in the z direction with $U(t)$ given by (47). The lump condition (45) can be satisfied with the choice

$$V=\boldsymbol{C}\cdot x+\tfrac{1}{2}m\nu_1^2(x_1^2+x_2^2)+\tfrac{1}{2}m\nu_3^2x_3^2+h, \tag{66}$$

and with the equations of motion

$$m(\ddot{a}_1-2\dot{a}_2\omega+\nu_1^2a_1)=D_1\cos(\omega t)+D_2\sin(\omega t)-C_1, \tag{67}$$

$$m(\ddot{a}_2+2\dot{a}_1\omega+\nu_1^2a_2)=-D_1\sin(\omega t)+D_2\cos(\omega t)-C_2, \tag{68}$$

$$m(\ddot{a}_3+\nu_3^2a_3)=D_3-C_3, \tag{69}$$

where $\boldsymbol{C}$, ν_1^2, ν_3^2, h, D_1, D_2 and D_3 are arbitrary real constants. If we further choose

$$f(t)=\boldsymbol{C}\cdot\boldsymbol{a}(t)+\tfrac{1}{2}m\nu_1^2(a_1^2+a_2^2)+\tfrac{1}{2}m\nu_3^2a_3^2(t), \tag{70}$$

the shape of the lump obeys the equation

$$-(\hbar^2/2m)\nabla^2_\xi(\boldsymbol{\xi})+[\boldsymbol{\xi}\cdot\boldsymbol{D}+\tfrac{1}{2}m(\omega^2+\nu_1^2)(\xi_1^2+\xi_2^2)+\tfrac{1}{2}m\nu_3^2\xi_3^2+h]R(\boldsymbol{\xi})=0. \tag{71}$$

Explicit solutions of (71) of the form $R_1(\xi_1)R_2(\xi_2)R_3(\xi_3)$ with each factor obeying an equation like (24) are easily obtained. Explicit solutions of (67)–(69) for $\nu_1\neq 0$ and $\nu_3\neq 0$ are

$$a_1(t)+\mathrm{i}a_2(t)=-\frac{C_1+\mathrm{i}C_2}{m\nu_1^2}+\mathrm{e}^{-\mathrm{i}\omega t}\bigg(\frac{D_1+\mathrm{i}D_2}{m(\omega^2+\nu_1^2)}+(F_1+\mathrm{i}F_2)\exp[\mathrm{i}t(\omega^2+\omega_1^2)^{1/2}]$$
$$+(G_1+\mathrm{i}G_2)\exp[-\mathrm{i}t(\omega^2+\nu_1^2)^{1/2}]\bigg) \tag{72}$$

and

$$a_3(t)=[(D_3-C_3)/m\nu_3^2]+F_3\cos(\nu_3 t)+G_3\sin(\nu_3 t), \tag{73}$$

where the F_i and G_i $(i=1,2,3)$ are arbitrary real constants, and a_1 and a_2 are respectively the real and imaginary parts of the right-hand side of equation (72). Note that for $\nu_3^2>0$ and $\nu_1^2+\omega^2>0$, the $a_i(t)$ are oscillatory functions, bounded for $|t|\to\infty$. This is not so for $\nu_1^2=\nu_3^2=0$ (the explicit expressions for that case are omitted here). In summary, we have exhibited a lump whose rotational motion is given by equation (47), translational motion by equations (72)–(73) and shape by equation (71).

6. Nonlinear Schrödinger equation

As an example of application of the method to nonlinear equations, consider the equation

$$\mathrm{i}\hbar\,\partial\psi/\partial t=[e\phi+V_0+(1/2m)(-\mathrm{i}\hbar\nabla-e\boldsymbol{A}/c)^2]\psi-k_0|\psi|^2\psi, \tag{74}$$

which differs from the Schrödinger equation (7) by the addition of the nonlinear term. This equation for the case $\phi=V_0=\boldsymbol{A}=0$ has been widely discussed (see e.g. Scott *et al* 1973, Ginibre and Velo 1980) because of its numerous physical applications in optics, plasma physics, superconductivity, particle physics etc.

Substituting for ψ the equation (4), equation (74) yields, as before, equations analogous to (8)–(13) with the sole difference that in equations (9) and (12)

$$(1/R)\nabla^2 R\to(1/R)\nabla^2 R+(2mk_0/\hbar^2)R^2. \tag{75}$$

In particular, the continuity equation and hence the relation between lump velocity and hydrodynamical velocity are unaltered. Further, since the change in (75) involves a function of $\boldsymbol{\xi}$ alone, for every potential which admits lump solutions in the linear case, there are lump solutions also in the nonlinear case, with the same form for the equations of motion for $a(t)$, and with the equation for the shape of the lump changed as in (75).

It may be useful to give explicit formulae for the case of one space dimension and $\phi=V_0=\boldsymbol{A}=0$. Seeking lumps of hydrodynamical velocity, i.e. $v_{\mathrm{lump}}=v$, we obtain

$$m\ddot{a}(t)=\frac{\hbar^2}{2m}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{R}\frac{\mathrm{d}^2R}{\mathrm{d}\xi^2}+\frac{2mk_0}{\hbar^2}R^2(\xi)\right)=C_3. \tag{76}$$

where C_3 is a real constant. Hence

$$R''(\xi)/R+(2mk_0/\hbar^2)R^2=(2m/\hbar^2)C_3\xi+C_4, \tag{77}$$

with C_4 real. For $C_3=0$ and $C_4>0$, an explicit solution is easily obtained using a result due to Reid (1971), that the equation

$$y''(\xi)-\omega^2 y(\xi)=C/y^{2p-1}, \tag{78}$$

has a solution (for $p\neq 0, 1$)

$$y(\xi)=\left(a^p\,\mathrm{e}^{p\omega\xi}+\frac{Ca^{-p}\,\mathrm{e}^{-p\omega\xi}}{4\omega^2(p-1)}\right)^{1/p}. \tag{79}$$

Thus, equation (77) for $C_3=0$ and $C_4>0$ has a solution

$$R(\xi)=a\left(\exp[-\xi(C_4)^{1/2}]+\frac{mk_0}{C_4}\left(\frac{a}{2\hbar}\right)^2\exp[\xi(C_4)^{1/2}]\right)^{-1}, \tag{80}$$

where a is an arbitrary real constant. Equation (80) may be rewritten

$$R(\xi)=\hbar(C_4/m|k_0|)^{1/2}\operatorname{sech}[C_4{}^{1/2}(\xi-\xi_0)], \qquad \text{if } k_0>0, \tag{81}$$

and

$$R(\xi)=\hbar(C_4/m|k_0|)^{1/2}\operatorname{cosech}[C_4{}^{1/2}(\xi-\xi_0)], \qquad \text{if } k_0<0, \tag{82}$$

where ξ_0 is an arbitrary real constant. The envelope soliton solution corresponding to (81) is well known (Scott *et al* 1973). Equation (82) also yields a lump; however it blows up at $\xi=\xi_0$.

The novel feature brought out by our method is the existence of lumps with non-zero acceleration corresponding to $C_3\neq 0$ in equation (77). Solutions of this equation for $k_0<0$ may be easily obtained by noticing that it reduces to the equation defining the second Painlevé transcendent (see e.g. Davis 1962)

$$\mathrm{d}^2y/\mathrm{d}x^2=3y^3+xy+\mu, \tag{83}$$

if we identify $\mu=0$, and

$$x=[(2m/\hbar^2)C_3\xi+C_4](\hbar^2/2mC_3)^{2/3}, \tag{84}$$

$$y=R(\xi)[2m/3(-k_0)]^{1/2}(2m\hbar C_3)^{-1/3}. \tag{85}$$

The shape of the lump $R(\xi)$ may now be easily computed using the extensive analytical results (appendix 4 of Davis (1962)) and numerical tables (table 2 of Davis (1962)) for the second Painlevé transcendent (Davis 1962, 1956).

We may also note that lumps of non-hydrodynamical velocity, e.g.

$$v=v_{\text{lump}}+b/R^2(\xi), \qquad v_{\text{lump}}=\dot a(t), \tag{86}$$

exist, with the equation of motion

$$m\ddot a(t)=C_3, \tag{87}$$

and the shape of the lump given by

$$\frac{R''(\xi)}{R}+\frac{2mk_0}{\hbar^2}R^2-\left(\frac{mb}{\hbar}\right)^2\frac{1}{R^4(\xi)}=\frac{2mC_3}{\hbar^2}\xi+C_4. \tag{88}$$

7. Concluding remarks

We have given in this paper a method to construct solutions which propagate without change of shape. In contrast to the usual solitonic solutions known in literature, which generally move with constant velocity, we also find solutions which describe lumps which have accelerated motion. Apart from linear acceleration the lump could be rotating as well. We believe that rotating lumps are a qualitatively new feature.

We have illustrated the method by finding envelope solitons for the linear, as well as nonlinear, Schrödinger equation. The interesting general result is that potentials which admit lump solutions in the linear case also have lumps in the nonlinear case, with the same motion $a(t)$ but with the shape of the lump depending on the nonlinearity.

This close relation does not appear mysterious if we recognise that (motivated by gauge invariance) we required the shape preserving property only for $|\psi|^2$. Even when ψ solves the linear Schrödinger equation, $|\psi|^2$ obeys the nonlinear hydrodynamical-type equations. The corresponding equations in the nonlinear Schrödinger equation case differ only in the differential equation for $R(\xi)$ due to the replacement (75). It is important to remark that this does not imply that in any lump solution $R(\xi)$ in the nonlinear case we can set $k_0 = 0$ to obtain the corresponding solution in the linear case. A counterexample is provided by the solutions (81) and (82) of equation (77) for $C_3 = 0$: the limit $k_0 = 0$ does not exist. The equations for $R(\xi)$ have to be solved separately in the linear and nonlinear cases.

What are the possible physical applications of the new lumps? For the linear Schrödinger equation the new square integrable lumps for the harmonic oscillator and for a charged particle in a constant magnetic field plus an axially symmetric electric field seem especially interesting. In particular, we have found (Roy and Singh 1981) that Sudarshan's diagonal coherent state representation for an arbitrary density operator for the harmonic oscillator in terms of the Schrödinger states, as well as other properties of coherent states useful in quantum optics (Glauber 1963a, b, c (1964), Sudarshan 1963, Klauder and Sudarshan 1968), can be generalised in terms of the new coherent states. Physical applications of the new solutions in the nonlinear case remain to be worked out.

Is there a systematic method to identify a proper analogue of multisoliton solutions? In the linear case if ψ_1 and ψ_2 are lump solutions, then $\psi_1 + \psi_2$ is also a solution; but in general $|\psi_1 + \psi_2|^2$ cannot be approximated by $|\psi_1|^2 + |\psi_2|^2$ even after a long time. For example, the coherent states of the oscillator always remain close to the centre of force. Hence $\psi_1 + \psi_2$ cannot in general be interpreted as a two-lump solution. In the case of localised lumps ψ_1 and ψ_2 whose separation $\to \infty$ for $t \to \infty$, $\psi_1 + \psi_2$ may be interpreted as a two-lump solution. We have not yet found for the linear Schrödinger equation square integrable lumps ψ_1, ψ_2 with this property (if we relaxed square integrability two solutions of the Berry–Balazs form equation (3) would do). We do not know whether the well known multisoliton solutions of nonlinear equations (Scott *et al* 1973) may be generalised to give multi-lump solutions.

The next major task is to apply our method to find new lump solutions of relativistic wave equations and classical field theories.

Note added in proof. We have become aware that the harmonic oscillator lumps (28) were known earlier (Senitzky 1954).

Appendix 1. Solutions of equations (30) for $C_1(t) = 0$, and $\dot{b}(t) \neq 0$

Since the left-hand side of equation (30) is of the form $(f_1(\xi) + f_2(t))$, using

$$\frac{\partial^2}{\partial\xi\,\partial t}(f_1(\xi) + f_2(t)) = 0, \tag{A1}$$

we obtain

$$b(t)\dot{b}(t)(u^2)''' + \ddot{b}u'' - u'\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\ddot{b}}{\dot{a}}\right) - (u^2)''\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{b\dot{b}}{\dot{a}}\right) = 0. \tag{A2}$$

Dividing by $2u'b\dot{b}$ ($u' \not\equiv 0$ for non-trivial lump), and again using (A1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\ddot{b}}{2b\dot{b}}\right)\left(\frac{u''}{u'}\right)' - \left(\frac{(u^2)''}{u'}\right)'\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(\mathrm{d}/\mathrm{d}t)(b\dot{b}/\dot{a})}{2b\dot{b}}\right) = 0. \tag{A3}$$

There are only three ways in which (A3) could be satisfied:

(i) $\left(\frac{u''}{u'}\right)' \neq 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(\mathrm{d}/\mathrm{d}t)(b\dot{b}/\dot{a})}{2b\dot{b}}\right) = 0,$

(ii) $\left(\frac{u''}{u'}\right)' \neq 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(\mathrm{d}/\mathrm{d}t)(b\dot{b}/\dot{a})}{2b\dot{b}}\right) \neq 0,$

(iii) $(u''/u')' = 0.$

We shall rule out the possibilities (i) and (ii) and show that (iii) can be realised only in the case of the Morse potential.

Case (i). It follows that $\ddot{b} = a_1 b^2 + a_3$, and $2b\dot{b} = \dot{a}(a_2 b^2 + a_4)$, where the a_i are constants. Using $b^2(t) \neq$ constant, equation (30) now yields

$$\tfrac{1}{2}(u^2)'' + a_1 u' - a_2[a_1 u + \tfrac{1}{2}(u^2)'] = a_6, \tag{A4}$$

$$(\hbar^2/2m^2)(R''/R)'' = a_3 u' - a_4[a_1 u + \tfrac{1}{2}(u^2)'] + a_7, \tag{A5}$$

$$a_6 b^2(t) - a_7 = \dddot{a}/\dot{a}, \tag{A6}$$

where

$$\begin{aligned}\dddot{a}/\dot{a} = (a_2 b^2(t) + a_4)^{-2}\{&2a_1(a_1 b^2 + a_3)(a_2 b^2 + a_4)^2 \\ &+ (a_1 a_4 - a_2 a_3)[(a_2 b^2 + a_4)(2a_1 b^2 - 2a_3) + 8a_4(a_1 b^2 + a_3)]\}.\end{aligned} \tag{A7}$$

Subcase $a_2 = 0$. Equations (A6) and (A7) yield

$$a_6 = 12a_1^2, \qquad a_7 = -8a_1 a_3.$$

Equation (A4) can be integrated to yield

$$[u - 3a_1(\xi + a_8/12a_1^2)]^3[u + 4a_1(\xi + a_8/12a_1^2)]^4 = a_9. \tag{A8}$$

Calculating the asymptotic behaviour of $u(\xi)$ for $\xi \to \infty$ from (A8), and using $u(\xi) \equiv R^{-2}(\xi)$, we can compute the asymptotic behaviour of $(R''/R)''$. This conflicts with the asymptotic behaviour calculated from (A5), showing the inconsistency of (A4)–(A7).

Subcase $a_2 \neq 0$. Equations (A6) and (A7) imply $a_1 a_4 - a_2 a_3 = 0$, $a_6 = 2a_1^2$ and $a_7 = -2a_1 a_3$. Equation (A4) can be integrated to yield

$$[\eta(v^2 - 2/a_2)]^{1/2} = a_{11}/\sqrt{a_2} + (2a_1/a_2)\ln[v + (v^2 - 2/a_2)^{1/2}] \tag{A9}$$

where

$$\eta \equiv \exp(a_2\xi + a_{10}), \qquad v \equiv (u + 2a_1/a_2)/\sqrt{\eta}. \tag{A10}$$

Again, the asymptotic behaviour of $u(\xi)$ following from above implies a behaviour of $(R''/R)''$ that conflicts with (A5).

This completes the proof of the inconsistency of case (i).

Case (ii). It follows from (A3) that

$$\frac{[(u^2)''/u']'}{(u''/u')'} = a_1 \tag{A11}$$

and

$$\dot{b} = a_1 b\dot{b}/\dot{a} + a_2 b^2 + a_3. \tag{A12}$$

Equation (A11) can be integrated to give

$$(u^2)' - a_1 u' = a_4 u + a_5. \tag{A13}$$

For $a_4 \neq 0$, this integrates to

$$u - (a_5/a_4 + \tfrac{1}{2}a_1)\ln(u + a_5/a_4) = \tfrac{1}{2}a_4(\xi + a_6). \tag{A14}$$

It is easy to show that for $\xi \to \infty$, equation (A.14) conflicts with equation (30). In the subcase $a_4 = 0$, (A13) is similarly seen to conflict with equation (30).

This completes the proof of the inconsistency of case (ii).

Case (iii). For $(u''/u')' = 0$, we must have $R(\xi)$ as given by equation (40). Hence $[(u^2)''/u']' \neq 0$. Hence (A3) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(\mathrm{d}/\mathrm{d}t)(b\dot{b}/\dot{a})}{2b\dot{b}}\right) = 0.$$

It is then easy to prove that equations (38)–(40) are necessary and sufficient for consistency of equation (30).

References

Berry M V and Balazs N L 1979 *Am. J. Phys.* **47** 264
de Broglie L 1927 in '*Electrons et Photons, Rapport au Ve Conseil Physique Solvay* 1930 (Paris: Gauthier-Villars) p 115
—— 1960 *Non-linear Wave Mechanics, a causal interpretation* (Amsterdam: Elsevier)
Davis H T 1956 *Studies in Differential Equations* (Evanston) pp 1–72
—— 1962 *Introduction to nonlinear differential and integral equations* (New York: Dover) p 239
Ginibre J and Velo G 1980 in *Nonlinear evolution equations and dynamical systems* (vol 120 *Lecture Notes in Physics*) ed M Boiti *et al* (Berlin: Springer)
Glauber R J 1963 *PRL* **10** 84
—— 1963b *Phys. Rev.* **130** 2529
—— 1963c *Phys. Rev.* **131** 2766
—— 1964 *Les Houches Lectures* in '*Quantum optics and Electronics*', ed C De Witt *et al* (New York: Gordon and Breach)
Ignatovich V K 1978 *Found. Phys.* **8** 565
Klauder J R and Sudarshan E C G 1968 *Fundamentals of Quantum Optics* (New York: Benjamin)
Lewis H R Jr 1967 *Phys. Rev. Lett.* **18** 510
Lorentz H A 1926 in letter to Schrödinger, 27 May, 1926, reprinted in *Letters on wave mechanics* 1967 (New York: Philosophical Library)
Madelung E 1926 *Z. Phys.* **40** 322
Messiah A 1961 *Quantum Mechanics* (Amsterdam: North-Holland) **1** 216–22
Nelson E 1967 *Dynamical Theories of Brownian Motion, Mathematical Notes* (Princeton, NJ: Princeton University Press)
Pinney E 1950 *Proc. Am. Math. Soc.* **1** 681

Reid J L 1971 *Proc. Am. Math. Soc.* **27** 61
Roy S M and Singh V 1981 *TIFR Preprint*
Schiff L I 1955 *Quantum Mechanics* (New York: McGraw-Hill) 2nd edn p 68
Schrödinger E 1926 reply to Lorentz, reprinted in *Letters on Wave Mechanics* 1967 (New York: Philosophical Library) and *Naturwiss.* **14** 664
Scott A, Chu F and McLaughlin D 1973 *Proc. IEEE* **61** 1443
Senitzky I R 1954 *Phys. Rev.* **95** 1115
Sudarshan E C G 1963 *PRL* **10** 277
Whittaker E T and Watson G N 1963 *A course of modern analysis* (Cambridge: CUP) 4th edn, p 347